Sampson Error

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1 Problem Definition of Fundamental Matrix Estimation

Here, we consider fundamental matrix estimaton from correspondences of two views or images. The point in image is represented by homogenous coordinate $\boldsymbol{x} = (u, v, 1)^T$. The epipolar constraint can be represented $\boldsymbol{x}'^T \boldsymbol{F} \boldsymbol{x} = 0$. Figure 1 shows the schematic of a point correspondence. The fundamental matrix $\boldsymbol{F}$ will be estimated from N correspondence $\{\boldsymbol{x}_i, \boldsymbol{x}'_i\}$

For estimation, we should take into account observation error. The i-th observed correspondence $(\boldsymbol{x}_i, \boldsymbol{x}'_i)$ include observation error, so that the epipolar constraint is broken. Namely, $\boldsymbol{x}'^T_i \boldsymbol{F} \boldsymbol{x}_i \neq 0$. Next, we consider perfect correspondence $(\hat{\boldsymbol{x}}_i, \hat{\boldsymbol{x}}'_i)$ for i-th observation. This perfect correspondence should satisfy the epipolar constraint, $\hat{\boldsymbol{x}}'^T_i \boldsymbol{F} \hat{\boldsymbol{x}}_i = 0$. The fundamental matrix estimation is performed by minimizing the distance between the observed correspondences and the perfect correspondences. The distance between the observed correspondence and the perfect correspondence is called *reprojection error*. The problem of the fundamental matrix estimation can be represented by

$$\min \sum_{i=1}^{N} ||\boldsymbol{x}_i - \hat{\boldsymbol{x}}_i||_2^2 + ||\boldsymbol{x}'_i - \hat{\boldsymbol{x}}'_i||_2^2 , \qquad \text{subject to} \quad \hat{\boldsymbol{x}}'^T_i \boldsymbol{F} \hat{\boldsymbol{x}}_i = 0 \,. \tag{1}$$

This problem can be generalized for general constraint as

$$\min \sum_{i=1}^{N} ||\boldsymbol{x}_i - \hat{\boldsymbol{x}}_i||_2^2 + ||\boldsymbol{x}'_i - \hat{\boldsymbol{x}}'_i||_2^2 , \qquad \text{subject to} \quad f(\boldsymbol{p}, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{x}}'_i) = 0 \,, \tag{2}$$

where the function $f()$ represents any constraints and p is the parameters. In the fundamental matrix estimation case, p is the elements of the fundamental matrix and $f(\boldsymbol{p}, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{x}}'_i) = \hat{\boldsymbol{x}}'^T_i \boldsymbol{F}(\boldsymbol{p}) \hat{\boldsymbol{x}}_i$.

2 Linear approximation and sampson error

The problem presented in Eq. (2) includes unknown variables $\hat{\boldsymbol{x}}_i$ and $\hat{\boldsymbol{x}}'_i$. If we could remove these unknown variables from the Eq. (2), the problem would be simplified. Applying the linear approximation to the constraints, we can remove the unknown variables and simplify the problem. The resultant approximated error is the sampson error. The sampson error is the approximated projectiton error with the linear approximation to the constraints.

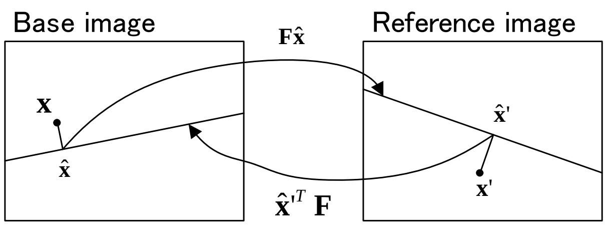

Figure 1: Example of epipolar geometry, where $\boldsymbol{x}$ and $\boldsymbol{x}'$ are observed points, $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{x}}'$ are the closest points of points which satisfy the epipolar constraint.

Now, let us derive the sampson error. The problem in Eq. (2) can be rewritten as

$$\min \sum_{i=1}^{N} ||\boldsymbol{\delta}_{\boldsymbol{X}_i}||_2^2 \,, \qquad \text{subject to} \quad f(\boldsymbol{p}, \boldsymbol{X}_i + \boldsymbol{\delta}_{\boldsymbol{X}_i}) = 0 \,, \tag{3}$$

where $\boldsymbol{\delta}_{\boldsymbol{X}_i} = \hat{\boldsymbol{X}}_i - \boldsymbol{X}_i$, $\boldsymbol{X}_i = (\boldsymbol{x}^T, \boldsymbol{x}_i'^T)^T$, and $\hat{\boldsymbol{X}}_i = (\hat{\boldsymbol{x}}_i^T, \hat{\boldsymbol{x}}_i'^T)^T$. The constraint can be approximated as

$$f(\boldsymbol{p}, \boldsymbol{X}_i + \boldsymbol{\delta}_{\boldsymbol{X}_i}) \simeq f(\boldsymbol{p}, \boldsymbol{X}_i) + \frac{\partial f}{\partial \boldsymbol{X}_i}(\boldsymbol{p}, \boldsymbol{X}_i)\, \boldsymbol{\delta}_{\boldsymbol{X}_i} \;=\; \varepsilon_i + \boldsymbol{J}_i \boldsymbol{\delta}_{\boldsymbol{X}_i} \;=\; 0\,. \tag{4}$$

By introducing the Lagrange multipliers λ_i, the problem can be rewritten as the optimization problem of

$$E \;=\; \sum_{i=1}^{N} E_i \,, \tag{5}$$

$$E_i \;=\; ||\boldsymbol{\delta}_{\boldsymbol{X}_i}||_2^2 - 2\lambda_i (\boldsymbol{J}_i \boldsymbol{\delta}_{\boldsymbol{X}_i} + \varepsilon_i)\,. \tag{6}$$

Taking derivative Eq. (6) with respect to $\boldsymbol{\delta}_{\boldsymbol{X}_i}$, we have

$$\frac{\partial E_i}{\partial \boldsymbol{\delta}_{\boldsymbol{X}_i}} \;=\; 2\boldsymbol{\delta}_{\boldsymbol{X}_i}^T - 2\lambda_i \boldsymbol{J}_i \;=\; 0 \tag{7}$$

From Eqs. (4) and (7), we obtain

$$\boldsymbol{J}_i \boldsymbol{J}_i^T \lambda_i \;=\; -\varepsilon_i\,. \tag{8}$$

Substituting λ_i derived from Eq. (8) to Eq. (7), $\boldsymbol{\delta}_{\boldsymbol{X}_i}$ can be expressed by

$$\boldsymbol{\delta}_{\boldsymbol{X}_i} \;=\; -\boldsymbol{J}^T (\boldsymbol{J}_i \boldsymbol{J}_i^T)^{-1} \varepsilon_i\,. \tag{9}$$

Finally, we have the Sampson error as the sum of the norm of $\boldsymbol{\delta}_{\boldsymbol{X}_i}$:

$$E(\boldsymbol{p}) \;=\; \sum_{i=1}^{N} ||\boldsymbol{\delta}_{\boldsymbol{X}_i}||_2^2 \;=\; \sum_{i=1}^{N} \varepsilon_i (\boldsymbol{J}_i \boldsymbol{J}_i^T)^{-1} \varepsilon_i \tag{10}$$

$$\;=\; \sum_{i=1}^{N} f(\boldsymbol{p}, \boldsymbol{X}_i) \left[\frac{\partial f}{\partial \boldsymbol{X}_i}(\boldsymbol{p}, \boldsymbol{X}_i) \frac{\partial f}{\partial \boldsymbol{X}_i}(\boldsymbol{p}, \boldsymbol{X}_i)^T \right]^{-1} f(\boldsymbol{p}, \boldsymbol{X}_i) \tag{11}$$

Acknowledgement

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References

[1] R. Hartley and A. Zisserman, Multiple View Geometry, Cambridge University Press, 2003.