Properties of Multivariate Gaussian Function

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1 Definition of multivariate Gaussian function [1]

Definition 1.1 *The multivariate Gaussian function* $G(\boldsymbol{x};\boldsymbol{\mu},\boldsymbol{\Sigma})$ *is defined by*

$$G(\boldsymbol{x};\boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{D/2}|\boldsymbol{\Sigma}|^{1/2}}\exp\left[\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right], \quad (1)$$

where $\boldsymbol{\mu}$ *is mean,* $\boldsymbol{\Sigma}$ *is covariance matrix, and* D *is dimension of* $\boldsymbol{x}$*.*

2 Product of multivariate Gaussian function

Lemma 2.1

$$\begin{aligned}(\boldsymbol{x}-\boldsymbol{\mu}_1)^T\boldsymbol{\Sigma}_1^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_1)+(\boldsymbol{x}-\boldsymbol{\mu}_2)^T\boldsymbol{\Sigma}_2^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_2) =& \\ (\boldsymbol{x}-\boldsymbol{m})^T\boldsymbol{S}^{-1}(\boldsymbol{x}-\boldsymbol{m})+(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)^T(\boldsymbol{\Sigma}_1+\boldsymbol{M}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)&, \quad (2)\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{S}^{-1} &= \boldsymbol{\Sigma}_1^{-1}+\boldsymbol{\Sigma}_2^{-1}, & (3)\\ \boldsymbol{m} &= \boldsymbol{S}(\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\mu}_1+\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\mu}_2), & (4)\\ \boldsymbol{M} &= \boldsymbol{\Sigma}_1+\boldsymbol{\Sigma}_2. & (5)\end{aligned}$$

Proof 2.1 *Assuming Eq. (2) is true, we will derive* $\boldsymbol{S}$*,* $\boldsymbol{m}$*, and* $\boldsymbol{M}$*. The left hand side of Eq. (2) is*

$$\begin{aligned}(\boldsymbol{x}-\boldsymbol{\mu}_1)^T\boldsymbol{\Sigma}_1^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_1)+(\boldsymbol{x}-\boldsymbol{\mu}_2)^T\boldsymbol{\Sigma}_2^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_2) =& \\ \boldsymbol{x}^T(\boldsymbol{\Sigma}_1^{-1}+\boldsymbol{\Sigma}_2^{-1})\boldsymbol{x}-2(\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\mu}_1+\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\mu}_1)^T\boldsymbol{x}\boldsymbol{\mu}_1^T\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\mu}_1+\boldsymbol{\mu}_2^T\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\mu}_2&. \quad (6)\end{aligned}$$

The right hand side of Eq. (2) is

$$\begin{aligned}(\boldsymbol{x}-\boldsymbol{m})^T\boldsymbol{S}^{-1}(\boldsymbol{x}-\boldsymbol{m})+(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)^T\boldsymbol{M}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2) =& \\ \boldsymbol{x}^T\boldsymbol{S}_1^{-1}\boldsymbol{x}-2(\boldsymbol{S}^{-1}\boldsymbol{m})^T\boldsymbol{x}+\boldsymbol{m}^T\boldsymbol{S}^{-1}\boldsymbol{m}+(\boldsymbol{\mu}_1-\boldsymbol{\mu}_1)^T\boldsymbol{M}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_1)&. \quad (7)\end{aligned}$$

Comparing coefficients of $\boldsymbol{x}$ *in Eqs. (6) and (7), we have*

$$\begin{aligned}\boldsymbol{S}^{-1} &= \boldsymbol{\Sigma}_1^{-1}+\boldsymbol{\Sigma}_2^{-1}, & (8)\\ \boldsymbol{S}^{-1}\boldsymbol{m} &= \boldsymbol{\Sigma}_1^{-1}\boldsymbol{\mu}_1+\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\mu}_2. & (9)\end{aligned}$$

Eq. (9) can be easily rewritten as

$$\boldsymbol{m} = \boldsymbol{S}(\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\mu}_1+\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\mu}_2). \quad (10)$$

The constant term in Eqs. (6) and (7) can be rewritten as

$$\begin{aligned}& \boldsymbol{\mu}_1^T\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\mu}_1+\boldsymbol{\mu}_2^T\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\mu}_2 \\ =\ & \boldsymbol{m}^T\boldsymbol{S}^{-1}\boldsymbol{m}+(\boldsymbol{\mu}_1-\boldsymbol{\mu}_1)^T\boldsymbol{M}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_1) \\ =\ & (\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\mu}_1+\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\mu}_2)^T\boldsymbol{S}(\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\mu}_1+\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\mu}_2)+(\boldsymbol{\mu}_1-\boldsymbol{\mu}_1)^T\boldsymbol{M}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_1) \\ =\ & \boldsymbol{\mu}_1^T(\boldsymbol{\Sigma}_1^{-1}\boldsymbol{S}\boldsymbol{\Sigma}_1^{-1}+\boldsymbol{M}^{-1})\boldsymbol{\mu}_1+\boldsymbol{\mu}_2^T(\boldsymbol{\Sigma}_2^{-1}\boldsymbol{S}\boldsymbol{\Sigma}_2^{-1}+\boldsymbol{M}^{-1})\boldsymbol{\mu}_2 \\ & +\boldsymbol{\mu}_1^T(\boldsymbol{\Sigma}_1^{-1}\boldsymbol{S}\boldsymbol{\Sigma}_2^{-1}-\boldsymbol{M}^{-1})\boldsymbol{\mu}_2^T+\boldsymbol{\mu}_2^T(\boldsymbol{\Sigma}_2^{-1}\boldsymbol{S}\boldsymbol{\Sigma}_1^{-1}-\boldsymbol{M}^{-1})\boldsymbol{\mu}_1^T. \quad (11)\end{aligned}$$

Comparing coefficients of $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ in Eq. (11), we have

$$\begin{aligned}
\boldsymbol{\Sigma}_1^{-1}\boldsymbol{S}\boldsymbol{\Sigma}_1^{-1} + \boldsymbol{M}^{-1} &= \boldsymbol{\Sigma}_1^{-1} & (12)\\
\boldsymbol{\Sigma}_2^{-1}\boldsymbol{S}\boldsymbol{\Sigma}_2^{-1} + \boldsymbol{M}^{-1} &= \boldsymbol{\Sigma}_2^{-1} & (13)\\
\boldsymbol{\Sigma}_1^{-1}\boldsymbol{S}\boldsymbol{\Sigma}_1^{-1} - \boldsymbol{M}^{-1} &= 0 & (14)\\
\boldsymbol{\Sigma}_2^{-1}\boldsymbol{S}\boldsymbol{\Sigma}_2^{-1} - \boldsymbol{M}^{-1} &= 0 & (15)
\end{aligned}$$

From Eq. (14), we have

$$\boldsymbol{M} = \boldsymbol{\Sigma}_2\boldsymbol{S}^{-1}\boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}_2(\boldsymbol{\Sigma}_1^{-1} + \boldsymbol{\Sigma}_2^{-1})\boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_2\,. \tag{16}$$

We can also derive same properties from Eq. (15). Eqs. (8), (10), and (16) show the lemma 2.1.

Proposition 2.2 *The product of two Gaussian functions $G(\boldsymbol{x};\boldsymbol{\mu}_1,\boldsymbol{\Sigma}_1)$ and $G(\boldsymbol{x};\boldsymbol{\mu}_2,\boldsymbol{\Sigma}_2)$ can be expressed as*

$$\begin{aligned}
&G(\boldsymbol{x};\boldsymbol{\mu}_1,\boldsymbol{\Sigma}_1) \times G(\boldsymbol{x};\boldsymbol{\mu}_2,\boldsymbol{\Sigma}_2)\\
&= \frac{|\boldsymbol{S}|^{1/2}}{(2\pi)^{D/2}|\boldsymbol{\Sigma}_1|^{1/2}|\boldsymbol{\Sigma}_2|^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)^T\boldsymbol{M}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)\right] G(\boldsymbol{x};\boldsymbol{m},\boldsymbol{S})\\
&= \frac{|\boldsymbol{S}|^{1/2}|\boldsymbol{M}|^{1/2}}{|\boldsymbol{\Sigma}_1|^{1/2}|\boldsymbol{\Sigma}_2|^{1/2}} G(\boldsymbol{0};\boldsymbol{\mu}_1-\boldsymbol{\mu}_2,\boldsymbol{M})\, G(\boldsymbol{x};\boldsymbol{m},\boldsymbol{S})\,. \qquad (17)
\end{aligned}$$

Proof 2.2 *We can easity derive the proposition using the lemma 2.1.*

Proposition 2.3 *The inner product between $G(\boldsymbol{x};\boldsymbol{\mu}_1,\boldsymbol{\Sigma}_1)$ and $G(\boldsymbol{x};\boldsymbol{\mu}_2,\boldsymbol{\Sigma}_2)$ can be calcuated by*

$$\begin{aligned}
< G(\boldsymbol{x};\boldsymbol{\mu}_1,\boldsymbol{\Sigma}_1), G(\boldsymbol{x};\boldsymbol{\mu}_2,\boldsymbol{\Sigma}_2) > &= \int G(\boldsymbol{x};\boldsymbol{\mu}_1,\boldsymbol{\Sigma}_1), G(\boldsymbol{x};\boldsymbol{\mu}_2,\boldsymbol{\Sigma}_2)\, d\boldsymbol{x}\\
&= \frac{|\boldsymbol{S}|^{1/2}}{(2\pi)^{D/2}|\boldsymbol{\Sigma}_1|^{1/2}|\boldsymbol{\Sigma}_2|^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)^T\boldsymbol{M}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)\right]\,.
\end{aligned} \tag{18}$$

3 KL-divergence of multivariate Gaussian function [1]

The KL-divergence of $p(x)$ with respect to $q(x)$ is defined by [3]

$$D_{KL}(p(x)|q(x)) = \int p(x) log \frac{p(x)}{q(x)} dx\,. \tag{19}$$

The KL-divergence of $G(\boldsymbol{x};\boldsymbol{\mu}_0,\boldsymbol{\Sigma}_0)$ with respect to $G(\boldsymbol{x};\boldsymbol{\mu}_1,\boldsymbol{\Sigma}_1)$ can be calculated by

$$D_{KL}(G(\boldsymbol{x};\boldsymbol{\mu}_0,\boldsymbol{\Sigma}_0)|G(\boldsymbol{x};\boldsymbol{\mu}_1,\boldsymbol{\Sigma}_1)) = \frac{1}{2}\left(\mathrm{tr}(\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\Sigma}_0) + (\boldsymbol{\mu}_1-\boldsymbol{\mu}_0)^T\boldsymbol{\Sigma}_1^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0) - \log\frac{|\boldsymbol{\Sigma}_0|}{|\boldsymbol{\Sigma}_1|} - D\right)\,. \tag{20}$$

4 Bhattacharyya and Hellinger distance of multivariate Gaussian function [2]

The Bhattacharyya coefficent between $p(x)$ and $q(x)$ is defined by [2]

$$BC(p,q) = \int \sqrt{p(x)q(x)}dx\,. \tag{21}$$

Using the Bhattacharyya coefficent $BC(p,q)$, the Bhattacharyya distance $D_B(p,q)$ and the Hellinger distance $D_H(p,q)$ can be defined by [2]

$$\begin{aligned}
D_B(p,q) &= -\log(BC(p,q))\,, & (22)\\
D_H(p,q) &= \sqrt{1-BC(p,q)}\,. & (23)
\end{aligned}$$

For two multivariate normal distribution of $G(\boldsymbol{x}; \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$ and $G(\boldsymbol{x}; \boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$,

$$BC(G(\boldsymbol{x}; \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0), G(\boldsymbol{x}; \boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)) = \exp\left(-\frac{1}{8}(\boldsymbol{\mu}_0 - \boldsymbol{\mu}_1)^T \boldsymbol{P}^{-1}(\boldsymbol{\mu}_0 - \boldsymbol{\mu}_1)\right) \sqrt{\frac{\sqrt{|\boldsymbol{\Sigma}_0|\,|\boldsymbol{\Sigma}_1|}}{|\boldsymbol{P}|}}, \tag{24}$$

$$D_B(G(\boldsymbol{x}; \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0), G(\boldsymbol{x}; \boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)) = \frac{1}{8}(\boldsymbol{\mu}_0 - \boldsymbol{\mu}_1)^T \boldsymbol{P}^{-1}(\boldsymbol{\mu}_0 - \boldsymbol{\mu}_1) + \frac{1}{2}\log\frac{|\boldsymbol{P}|}{\sqrt{|\boldsymbol{\Sigma}_0|\,|\boldsymbol{\Sigma}_1|}}, \tag{25}$$

where

$$\boldsymbol{P} = \frac{\boldsymbol{\Sigma}_0 + \boldsymbol{\Sigma}_1}{2} \tag{26}$$

References

[1] http://en.wikipedia.org/wiki/Multivariate_normal_distribution

[2] http://en.wikipedia.org/wiki/Bhattacharyya_distance

[3] http://en.wikipedia.org/wiki/KL_divergence